TS 1 IRIS : Transformation de Laplace Sujet A

On rappelle que le signal unité est défini par : $\mathcal{U} : \begin{cases} \mathcal{U}(t) = 0 & \text{si } t < 0 \\ \mathcal{U}(t) = 1 & \text{si } t \geqslant 0 \end{cases}$

1) Laplace

Calculer les transformées de Laplace suivantes :

a) $\mathcal{L}\left[\Big(t + 1 + 5\sin(3t)\Big)\,\mathcal{U}(t)\right]$

b) $\mathcal{L}\left[\left(\dfrac{t}{2} + 1\right) e^{-t}\,\mathcal{U}(t) + (t+2)\,\mathcal{U}(t-2)\right]$

c) $\mathcal{L}\left[\cos\left(\dfrac{t}{3}\right) e^{-2t}\,\mathcal{U}(t)\right]$

2) Laplace inverse

Calculer les originaux suivants :

a) $\mathcal{L}^{-1}\left[\dfrac{5}{p} - \dfrac{1}{p^3} + \dfrac{3}{p+1}\right]$

b) $\mathcal{L}^{-1}\left[\dfrac{p+2}{p\,(p+5)} + \dfrac{5}{p+1}\right]$

c) $\mathcal{L}^{-1}\left[\dfrac{p-3}{(p^2+1)} + \dfrac{e^{-p}}{p+5}\right]$

3) Équations différentielles

Utiliser la transformée de Laplace pour déterminer la solution particulière de chacune des équations différentielles suivantes :

a) $x''(t) + 5x'(t) + 6x(t) = \mathcal{U}(t)$ conditions initiales : $\begin{cases} x(0) = 0 \\ x'(0) = 1 \end{cases}$

b) $x''(t) + 6x'(t) + 9x(t) = e^{-2t}\,\mathcal{U}(t)$ conditions initiales : $\begin{cases} x(0) = 0 \\ x'(0) = 0 \end{cases}$

c) $x''(t) + 4x(t) = 2\,\mathcal{U}(t)$ conditions initiales : $\begin{cases} x(0) = 0 \\ x'(0) = 1 \end{cases}$

d) $x''(t) + 2x'(t) + 2x(t) = 0$ conditions initiales : $\begin{cases} x(0) = 1 \\ x'(0) = 1 \end{cases}$

TS 1 IRIS : Transformation de Laplace Sujet B

On rappelle que le signal unité est défini par : $\mathcal{U} : \begin{cases} \mathcal{U}(t) = 0 & \text{si } t < 0 \\ \mathcal{U}(t) = 1 & \text{si } t \geqslant 0 \end{cases}$

1) Laplace

Calculer les transformées de Laplace suivantes :

a) $\mathcal{L}\left[\Big(t - 1 + 3\sin(2t)\Big)\,\mathcal{U}(t)\right]$

b) $\mathcal{L}\left[\left(\dfrac{t}{3} + 2\right) e^{-t}\,\mathcal{U}(t) + (t+1)\,\mathcal{U}(t-3)\right]$

c) $\mathcal{L}\left[\cos\left(\dfrac{t}{2}\right) e^{-3t}\,\mathcal{U}(t)\right]$

2) Laplace inverse

Calculer les originaux suivants :

a) $\mathcal{L}^{-1}\left[\dfrac{1}{p} - \dfrac{2}{p^3} + \dfrac{2}{p+5}\right]$

b) $\mathcal{L}^{-1}\left[\dfrac{p+1}{p\,(p+2)} + \dfrac{3}{p+2}\right]$

c) $\mathcal{L}^{-1}\left[\dfrac{p-5}{(p^2+1)} + \dfrac{e^{-p}}{p+3}\right]$

3) Équations différentielles

Utiliser la transformée de Laplace pour déterminer la solution particulière de chacune des équations différentielles suivantes :

a) $x''(t) + 6x'(t) + 8x(t) = \mathcal{U}(t)$ conditions initiales : $\begin{cases} x(0) = 0 \\ x'(0) = 1 \end{cases}$

b) $x''(t) + 4x'(t) + 4x(t) = e^{-3t}\,\mathcal{U}(t)$ conditions initiales : $\begin{cases} x(0) = 0 \\ x'(0) = 0 \end{cases}$

c) $x''(t) + 4x(t) = 2\,\mathcal{U}(t)$ conditions initiales : $\begin{cases} x(0) = 0 \\ x'(0) = 1 \end{cases}$

d) $x''(t) + 2x'(t) + 2x(t) = 0$ conditions initiales : $\begin{cases} x(0) = 1 \\ x'(0) = 1 \end{cases}$

TS 1 IRIS : Transformation de Laplace A (Solutions)

1) Laplace

a) $$\boxed{\mathcal{L}\left[\Big(t+1+5\sin(3t)\Big)\,\mathcal{U}(t)\right]=\frac{1}{p^2}+\frac{1}{p}+\frac{15}{p^2+9}}$$

b) $$\mathcal{L}\left[\left(\frac{t}{2}+1\right)e^{-t}\,\mathcal{U}(t)+(t+2)\,\mathcal{U}(t-2)\right]$$

$$\begin{aligned}
\mathcal{L}\left[\left(\frac{t}{2}+1\right)\,\mathcal{U}(t)\right]&=\frac{1}{2p^2}+\frac{1}{p}\\
\mathcal{L}\left[\left(\frac{t}{2}+1\right)e^{-t}\,\mathcal{U}(t)\right]&=\frac{1}{2(p+1)^2}+\frac{1}{p+1}\\
\mathcal{L}\left[\Big(t+2\Big)\,\mathcal{U}(t-2)\right]&=\mathcal{L}\left[\Big((t-2)+4\Big)\,\mathcal{U}(t-2)\right]\\
&=\left(\frac{1}{p^2}+\frac{4}{p}\right)e^{-2p}
\end{aligned}$$

$$\boxed{\mathcal{L}\left[\left(\frac{t}{2}+1\right)e^{-t}\,\mathcal{U}(t)+(t+2)\,\mathcal{U}(t-2)\right]=\frac{1}{2(p+1)^2}+\frac{1}{p+1}+\left(\frac{1}{p^2}+\frac{4}{p}\right)e^{-2p}}$$

c) $$\mathcal{L}\left[\cos\left(\frac{t}{3}\right)e^{-2t}\,\mathcal{U}(t)\right]$$

$$\begin{aligned}
\mathcal{L}\left[\cos\left(\frac{t}{3}\right)\,\mathcal{U}(t)\right]&=\frac{p}{p^2+(\frac{1}{3})^2}=\frac{9p}{9p^2+1}\\
\mathcal{L}\left[\cos\left(\frac{t}{3}\right)e^{-2t}\,\mathcal{U}(t)\right]&=\frac{9(p+2)}{9(p+2)^2+1}
\end{aligned}$$

$$\boxed{\mathcal{L}\left[\cos\left(\frac{t}{3}\right)e^{-2t}\,\mathcal{U}(t)\right]=\frac{9p+18}{9p^2+36p+37}}$$

2) Laplace inverse

a) $$\boxed{\mathcal{L}^{-1}\left[\frac{5}{p}-\frac{1}{p^3}+\frac{3}{p+1}\right]=\left(5-\frac{1}{2}t^2+3e^{-t}\right)\,\mathcal{U}(t)}$$

b) $$\mathcal{L}^{-1}\left[\frac{p+2}{p\,(p+5)}+\frac{5}{p+1}\right]$$

$$\frac{p+2}{p\,(p+5)}+\frac{5}{p+1}=\frac{\frac{2}{5}}{p}+\frac{\frac{3}{5}}{p+5}+\frac{5}{p+1}$$

$$\boxed{\mathcal{L}^{-1}\left[\frac{p+2}{p\,(p+5)}+\frac{5}{p+1}\right]=\frac{1}{5}\left(2+3e^{-5t}+25e^{-t}\right)\,\mathcal{U}(t)}$$

c) $\mathcal{L}^{-1}\left[\dfrac{p-3}{(p^2+1)}+\dfrac{e^{-p}}{p+5}\right]$

$$\frac{p-3}{(p^2+1)}+\frac{e^{-p}}{p+5}=\frac{p}{(p^2+1)}-\frac{3}{(p^2+1)}+\frac{1}{p+5}e^{-p}$$

$$\boxed{\mathcal{L}^{-1}\left[\frac{p-3}{(p^2+1)}+\frac{e^{-p}}{p+5}\right]=\big(\cos(t)-3\sin(t)\big)\,\mathcal{U}(t)+e^{-5(t-1)}\,\mathcal{U}(t-1)}$$

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3) Équations différentielles

On notera : $\mathcal{L}\,[x]=X$

a) $x''(t)+5x'(t)+6x(t)=\mathcal{U}(t)$ conditions initiales : $\left\{\begin{array}{lcl} x(0) & = & 0\\ x'(0) & = & 1\end{array}\right.$

$$\begin{aligned}
p^2X(p)-1+5p\,X(p)+6\,X(p)&=\frac{1}{p}\\
(p^2+5\,p+6)X(p)&=\frac{1}{p}+1\\
(p+2)(p+3)X(p)&=\frac{p+1}{p}\\
X(p)&=\frac{p+1}{p(p+2)(p+3)}\\
&=\frac{\frac{1}{6}}{p}+\frac{\frac{1}{2}}{p+2}-\frac{\frac{2}{3}}{p+3}
\end{aligned}$$

$$\boxed{x(t)=\left(\frac{1}{6}+\frac{1}{2}e^{-2t}-\frac{2}{3t}e^{-3}\right)\,\mathcal{U}(t)=\frac{1}{6}\big(1+3e^{-2t}-4e^{-3t}\big)\,\mathcal{U}(t)}$$

b) $x''(t)+6x'(t)+9x(t)=e^{-2t}\,\mathcal{U}(t)$ conditions initiales : $\left\{\begin{array}{lcl} x(0) & = & 0\\ x'(0) & = & 0\end{array}\right.$

$$\begin{aligned}
p^2X(p)+6p\,X(p)+9\,X(p)&=\frac{1}{p+2}\\
(p^2+6p+9)X(p)&=\frac{1}{p+2}\\
(p+3)^2X(p)&=\frac{1}{p+2}\\
X(p)&=\frac{1}{(p+1)(p+3)^2}\\
&=\frac{\frac{1}{4}}{p+1}-\frac{\frac{1}{2}}{(p+3)^2}-\frac{\frac{1}{4}}{p+3}
\end{aligned}$$

$$\boxed{x(t)=\left(\frac{1}{4}e^{-t}-\frac{1}{2}t\,e^{-3t}-\frac{1}{4}e^{-3t}\right)\,\mathcal{U}(t)=\frac{1}{4}\big(e^{-t}-(2t+1)e^{-3t}\big)\,\mathcal{U}(t)}$$

c) $x''(t) + 4x(t) = 2\,\mathcal{U}(t)$ conditions initiales : $\left\{\begin{array}{lcl} x(0) & = & 0 \\ x'(0) & = & 1 \end{array}\right.$

$$\begin{aligned}
p^2X(p) - 1 + 4\,X(p) &= \frac{2}{p} \\
(p^2+4)X(p) &= \frac{2}{p} + 1 \\
(p^2+4)X(p) &= \frac{p+2}{p} \\
X(p) &= \frac{p+2}{p(p^2+4)} \\
&= \frac{\frac{1}{2}}{p} + \frac{-\frac{1}{2}p+1}{p^2+4} \\
&= \frac{1}{2}\left(\frac{1}{p} - \frac{p}{p^2+4} + \frac{2}{p^2+4}\right)
\end{aligned}$$

$$\boxed{x(t) = \frac{1}{2}\big(1 - \cos(2t) + 2\sin(2t)\big)\,\mathcal{U}(t)}$$

d) $x''(t) + 2x'(t) + 2x(t) = 0$ conditions initiales : $\left\{\begin{array}{lcl} x(0) & = & 1 \\ x'(0) & = & 1 \end{array}\right.$

$$\begin{aligned}
p^2X(p) - p - 1 + 2\big(pX(p) - 1\big) + 2\,X(p) &= 0 \\
(p^2+2p+2)X(p) &= p+3 \\
X(p) &= \frac{p+3}{p^2+2p+2} \\
&= \frac{p+3}{(p+1)^2+1} \\
&= \frac{p+1}{(p+1)^2+1} + \frac{2}{(p+1)^2+1}
\end{aligned}$$

$$\boxed{x(t) = \Big(\cos(t) + 2\sin(t)\Big)\,e^{-t}\;\mathcal{U}(t)}$$

TS 1 IRIS : Transformation de Laplace B (Solutions)

1) Laplace

a) $$\boxed{\mathcal{L}\left[\Big(t-1+3\sin(2t)\Big)\,\mathcal{U}(t)\right]=\frac{1}{p^2}-\frac{1}{p}+\frac{6}{p^2+4}}$$

b) $\mathcal{L}\left[\left(\frac{t}{3}+2\right)e^{-t}\,\mathcal{U}(t)+(t+1)\,\mathcal{U}(t-3)\right]$

$$\begin{aligned}
\mathcal{L}\left[\left(\frac{t}{3}+2\right)\,\mathcal{U}(t)\right]&=\frac{1}{3p^2}+\frac{2}{p}\\
\mathcal{L}\left[\left(\frac{t}{3}+2\right)e^{-t}\,\mathcal{U}(t)\right]&=\frac{1}{3(p+1)^2}+\frac{2}{p+1}\\
\mathcal{L}\left[\Big(t+1\Big)\,\mathcal{U}(t-3)\right]&=\mathcal{L}\left[\Big((t-3)+4\Big)\,\mathcal{U}(t-3)\right]\\
&=\left(\frac{1}{p^2}+\frac{4}{p}\right)e^{-3p}
\end{aligned}$$

$$\boxed{\mathcal{L}\left[\left(\frac{t}{3}+2\right)e^{-t}\,\mathcal{U}(t)+(t+1)\,\mathcal{U}(t-3)\right]=\frac{1}{3(p+1)^2}+\frac{2}{p+1}+\left(\frac{1}{p^2}+\frac{4}{p}\right)e^{-3p}}$$

c) $\mathcal{L}\left[\cos\left(\frac{t}{2}\right)e^{-3t}\,\mathcal{U}(t)\right]$

$$\begin{aligned}
\mathcal{L}\left[\cos\left(\frac{t}{2}\right)\,\mathcal{U}(t)\right]&=\frac{p}{p^2+(\frac{1}{2})^2}=\frac{4p}{4p^2+1}\\
\mathcal{L}\left[\cos\left(\frac{t}{2}\right)e^{-3t}\,\mathcal{U}(t)\right]&=\frac{4(p+3)}{4(p+3)^2+1}
\end{aligned}$$

$$\boxed{\mathcal{L}\left[\cos\left(\frac{t}{3}\right)e^{-2t}\,\mathcal{U}(t)\right]=\frac{4p+12}{4p^2+24p+36}}$$

2) Laplace inverse

a) $$\boxed{\mathcal{L}^{-1}\left[\frac{1}{p}-\frac{2}{p^3}+\frac{2}{p+5}\right]=\left(1-t^2+2e^{-5t}\right)\mathcal{U}(t)}$$

b) $\mathcal{L}^{-1}\left[\frac{p+1}{p\,(p+2)}+\frac{3}{p+2}\right]$

$$\frac{p+1}{p\,(p+2)}+\frac{3}{p+2}=\frac{\frac{1}{2}}{p}+\frac{\frac{1}{2}}{p+2}+\frac{3}{p+2}=\frac{\frac{1}{2}}{p}+\frac{\frac{7}{2}}{p+2}$$

$$\boxed{\mathcal{L}^{-1}\left[\frac{p+2}{p\,(p+5)}+\frac{5}{p+1}\right]=\frac{1}{2}\left(1+7e^{-2t}\right)\,\mathcal{U}(t)}$$

c) $\mathcal{L}^{-1}\left[\dfrac{p-5}{(p^2+1)}+\dfrac{e^{-p}}{p+3}\right]$

$$\frac{p-5}{(p^2+1)}+\frac{e^{-p}}{p+3}=\frac{p}{(p^2+1)}-\frac{5}{(p^2+1)}+\frac{1}{p+3}e^{-p}$$

$$\boxed{\mathcal{L}^{-1}\left[\frac{p-5}{(p^2+1)}+\frac{e^{-p}}{p+3}\right]=\big(\cos(t)-5\sin(t)\big)\,\mathcal{U}(t)+e^{-3(t-1)}\,\mathcal{U}(t-1)}$$

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3) Équations différentielles

On notera : $\mathcal{L}\,[x]=X$

a) $x''(t)+6x'(t)+8x(t)=\mathcal{U}(t)$ conditions initiales : $\left\{\begin{array}{rcl} x(0) & = & 0 \\ x'(0) & = & 1 \end{array}\right.$

$$\begin{aligned}
p^2X(p)-1+6p\,X(p)+8\,X(p) &= \frac{1}{p}\\
(p^2+6\,p+8)X(p) &= \frac{1}{p}+1\\
(p+2)(p+4)X(p) &= \frac{p+1}{p}\\
X(p) &= \frac{p+1}{p(p+2)(p+4)}\\
&= \frac{\frac{1}{8}}{p}+\frac{\frac{1}{4}}{p+2}-\frac{\frac{3}{8}}{p+4}
\end{aligned}$$

$$\boxed{x(t)=\left(\frac{1}{8}+\frac{1}{4}e^{-2t}-\frac{3}{8}e^{-4t}\right)\,\mathcal{U}(t)=\frac{1}{8}\big(1+2e^{-2t}-3e^{-4t}\big)\,\mathcal{U}(t)}$$

b) $x''(t)+4x'(t)+4x(t)=e^{-3t}\,\mathcal{U}(t)$ conditions initiales : $\left\{\begin{array}{rcl} x(0) & = & 0 \\ x'(0) & = & 0 \end{array}\right.$

$$\begin{aligned}
p^2X(p)+4p\,X(p)+4\,X(p) &= \frac{1}{p+3}\\
(p^2+4p+4)X(p) &= \frac{1}{p+3}\\
(p+2)^2X(p) &= \frac{1}{p+3}\\
X(p) &= \frac{1}{(p+3)(p+2)^2}\\
&= \frac{1}{p+3}-\frac{1}{(p+2)^2}-\frac{1}{p+2}
\end{aligned}$$

$$\boxed{x(t)=\big(e^{-3t}-t\,e^{-2t}-e^{-2t}\big)\,\mathcal{U}(t)=\big(e^{-3t}-(t+1)e^{-2t}\big)\,\mathcal{U}(t)}$$

c) $x''(t) + 4x(t) = 2\,\mathcal{U}(t)$ conditions initiales : $\left\{\begin{array}{rcl} x(0) & = & 0 \\ x'(0) & = & 1 \end{array}\right.$

$$\begin{aligned} p^2 X(p) - 1 + 4\,X(p) &= \frac{2}{p} \\ (p^2+4)X(p) &= \frac{2}{p} + 1 \\ (p^2+4)X(p) &= \frac{p+2}{p} \\ X(p) &= \frac{p+2}{p(p^2+4)} \\ &= \frac{\frac{1}{2}}{p} + \frac{-\frac{1}{2}p+1}{p^2+4} \\ &= \frac{1}{2}\left(\frac{1}{p} - \frac{p}{p^2+4} + \frac{2}{p^2+4}\right) \end{aligned}$$

$$\boxed{x(t) = \frac{1}{2}\big(1 - \cos(2t) + 2\sin(2t)\big)\,\mathcal{U}(t)}$$

d) $x''(t) + 2x'(t) + 2x(t) = 0$ conditions initiales : $\left\{\begin{array}{rcl} x(0) & = & 1 \\ x'(0) & = & 1 \end{array}\right.$

$$\begin{aligned} p^2 X(p) - p - 1 + 2\big(pX(p) - 1\big) + 2\,X(p) &= 0 \\ (p^2+2p+2)X(p) &= p+3 \\ X(p) &= \frac{p+3}{p^2+2p+2} \\ &= \frac{p+3}{(p+1)^2+1} \\ &= \frac{p+1}{(p+1)^2+1} + \frac{2}{(p+1)^2+1} \end{aligned}$$

$$\boxed{x(t) = \Big(\cos(t) + 2\sin(t)\Big)\,e^{-t}\ \mathcal{U}(t)}$$